Electrodynamics

Riley Moriss

December 31, 2025

I like the way the [Gri13] summarises the goals of theory.

> The fundamental problem electrodynamics hopes to solve is this: We have some electric charges, $q_1, q_2, q_3, ...$ (source charges); what force do they exert on another charge, Q (test charge)? The positions of the source charges are given (as functions of time); the trajectory of the test particle is to be calculated.

He starts also with one axiom or experimental observation "the principle of superposition" that is the total force on the test charge is simply the sum of the forces exerted by each of the source charges, there is no interaction. The force on Q from q

> depends on the separation distance r between the charges, it also depends on both their velocities and on the acceleration of q. Moreover, it is not the position, velocity, and acceleration of q right now that matter: electromagnetic "news" travels at the speed of light, so what concerns Q is the position, velocity, and acceleration q had at some earlier time, when the message left.

there is a formulation of this in generality, however the goal is to develop some simpler ideas first and then arrive there.

1 Electrostatics

The first situation to deal with is when the source charges are all stationary. In this situation the answer is completely known and easy to state, the force on Q at position $r_1 \in \mathbb{R}^3$ exerted by q at position $r_2 \in \mathbb{R}^3$, then if we denote $\mathbf{r} = r_1 - r_2$

$$F = \frac{qQ}{|\mathbf{r}|^2}\hat{\mathbf{r}}.$$

And thats it. Thats electrostatics. The rest is devloping tools to make solving the equations simpler.

If we have $q_1, ..., q_n$ source charges each at distance r_i from our test charge then the force on our test charge is

$$F = Q\left(\sum_i \frac{q_i}{r_i^2}\hat{\mathbf{r}}_i\right) = QE(\mathbf{r})$$

the quantity $E(\mathbf{r})$ doesnt depend on the size of the test charge, but only the position. At each point in $\mathbb{R}^3$ it assigns another vector in $\mathbb{R}^3$ and so E is a vector field. The most obvious interpretation is that this vector field is the force per unit charge that would be exerted on a test charge at the point.

One thing that can help solve for this field is the differential or integral equations given by the divergence and curl of this vector field. The first is "Gausses law" which states that E must satisfy for any enclosed surface S

$$\int_S E \cdot \hat{n}_S \mathrm{ds} = Q_{\mathrm{enc}}$$

where Q_{enc} is the total charge enclosed by the surface. Recall that $\hat{n}_S$ is the normal vector to the surface, as a function of s. This can be very useful when the magnitude of the force is constant through the surface, in which case it can be taken out of the integral.

Another thing that can be shown is that the the line integral along any closed path is zero and hence the curl of the vector field is zero

$$\nabla \times E = 0$$

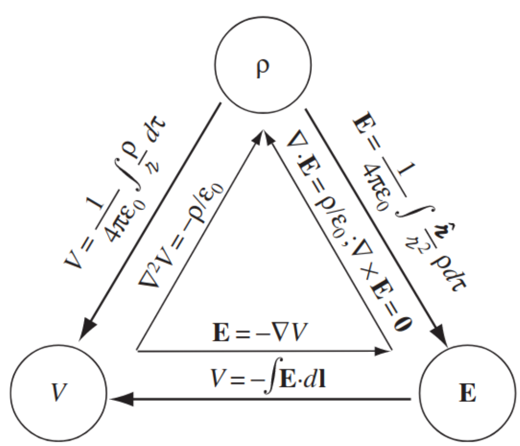
It is a theorem of vector calculus that any so called "conservative" vector field is the derivative of a scalar field, called the potential, hence there exists some $V : \mathbb{R}^3 \to \mathbb{R}$ such that

$$E = \nabla V$$

Note that since E is what we care about V is only defined up to a constant. The point of introducing V is that if we consider a charge that is not point like, as say an approximation to larger bodies, then its charge is described by a charge density $\rho : \mathbb{R}^3 \to \mathbb{R}$ and the potential satisfies

$$\nabla^2 V = \rho$$

The relations between these fundamental quantites E, ρ, V are summarised by the diagram



2 Magnets

Here the field concept becomes a little more real. Magnetism or magnetic fields feel like a much more intrinsically field like object. The reason for this is because of the experimental fact that all magnetic fields arise as the *motion* of electric charge or fields (even in rocks it is the motion of the electrons in their orbits). If we are given an already existent magnetic field, that is a vector field $B : \mathbb{R}^3 \to \mathbb{R}^3$, then it exerts a force on a charged particle moving at velocity v with charge Q at point $r \in \mathbb{R}^3$ given by

$$F = Q(v \times B(r))$$

That is the force is the magnitude of the magnetic field scaled by the charge and in the direction of the "right hand rule", that is perpendicular to the direction of motion.

If we have a charge density that is in motion then the current density is given by

$$J = \rho v$$

where v is the velocity vector (field) of the charge density (fluid). If this charge density is moving in a given magnetic field then its force experienced is

$$F = \int J \times B \mathrm{d}r$$

I guess; it is not clear. This current density satisfies some differential equations, that we probably wont write down.

2.1 Magnostatics

So a moving charge in a magnetic field experiences as force but, a moving charge also creates a magnetic field. The shape and size of this field is more complicated than determining the force on a test particle. Steady currents produce constant magnetic fields. This is the content of magnostatics. In this context the magnetic field is described by the Biot-Savart law when the current is along a line

$$B(r) = \int \frac{J \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} \mathrm{d}l$$

where l is the path of the flow of the line current. Note that the superposition principle applies to magnetic fields as well, so it is sufficient to determine those coming from single wires and then add them up, or in the case of an area to take an integral (one presumes). Thus to some extent this provides the full answer (at least for steady currents). The general case for volume currents is indeed just

$$B(r) = \int \frac{J(r') \times \hat{\mathbf{r}}}{|\mathbf{r}|^2} \mathrm{d}r'$$

Here recal that the bold $\mathbf{r}$ is the difference $r - r'$ and we integrate over r', that is over all space (by fubinis theorem this is exactly as we described).

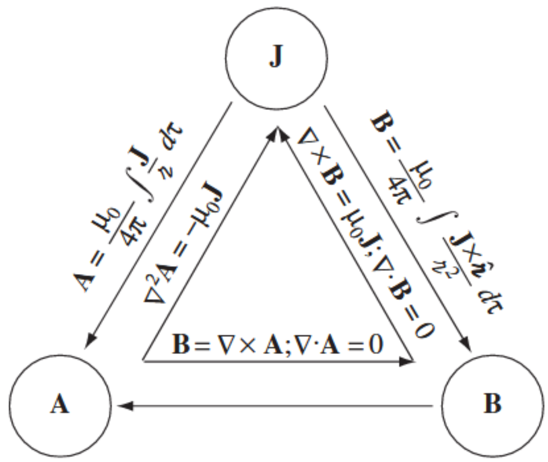
For steady state currents one can compute the curl of this B above as get that

$$\nabla \times B = J$$

which is called Ampreres law. Its divergence can also be shown to be zero $\nabla \cdot B = 0$. By vector calculus this implies that B is the divergence of a vector potential A, that is there exists an A such that

$$B = \nabla \times A$$

This A is only defined up to the addition of a function with zero curl, we use this to normalise A to have zero divergence. "Since magnetic forces do no work, A does not admit a simple physical interpretation in terms of potential energy per unit charge". Griffiths summarises



3 Maxwells Equations

Finally we need to abandon the static assumptions. The key is that a changing magnetic field produces an electric field and vice versa. Thus we amend the electrostatic and magnostatic differential equations of zero div and curl to contain terms from *the other*. To make this explicit we have the following set of differential equations

$$\nabla \cdot E = \rho, \quad \nabla \cdot B = 0$$

$$\nabla \times E = \frac{-\partial B}{\partial t}, \quad \nabla \times B = J + \frac{\partial E}{\partial t}$$

which tell you how to find the fields from some arrangement of charges (they depend on ρ and J) and the "force law" which tells you how to find the force on charges given the fields

$$F = q(E + v \times B).$$

Thus the original question can be answered, we have the initial source charges that are possibly moving around in some given way, the first set of equations tells us the magnetic and electric fields they produce. Then when we introduce our test particle the second equation tells us its motion, using Newtons second law.

Remark. [Gri13, 8.2.1] points out that Newtons third law is incompatable with electrodynamics. It is with electrostatics.

Remark. Landau and Lifshitz gives an action formulation of electrodynamics, so this is not the only formulation. It is not clear to what extent these equations are either necissary or sufficient, however they are by experimental reasons both I guess (in a paradigm which means neither).

Remark. We are assuming that like there is a single "test" particle and everything else is "nailed down". But this is never the case. Thus nothing in these books tells you how to really describe even a two body system where the two particles are interacting with one another. I guess this is a very hard problem and these methods provide good approximations for large experiments where the aparatus is "nailed down" by much larger forces. On the other hand wikipedias two body problem page has exact solutions in simply regimes, but its methods are to reduce to one body problems, so I guess this also furnishes the solution in that case too.

Remark. It is not clear from these equations that we are taking into account that the information is travelling at the speed of light.

4 Properties of the Field

Maxwells equations entail that charge is not just globaly conserved, but also locally. This is the idea that if the charge in a bounded region changes then that amount of charge must have passed through the bounding surface. This is mathematically stated as

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot J$$

4.1 Energy

If we have a configuration of stationary charges then the work required to move a charge Q from point A to point B is

$$W = \int_A^B F \cdot d\ell = -Q \int_A^B E \cdot d\ell = Q[V(B) - V(A)]$$

the integral along the path taken. For a volume charge density the amount of work required to assemble the "infinitesimal" charges into a given density ρ is

$$W = \int \rho V d\tau$$

where τ is integrated over all space (if no density is at some point then it doesnt contribute to the integral and therefore the work). Using Maxwells equations we can get that this is

$$W = \int (E \cdot E) d\tau$$

where E is the field of the assembled charge.

In magnostatics we have a similar analysis that tells us that the work done to produce a current and therefore produce the magnetic field B is given by

$$W = \int (B \cdot B) d\tau$$

This establishes a correspondence between energy and fields, namely the energy required to make some electromagnetic field is given by

$$\int E^2 + B^2 d\tau$$

or in other words we can say that the field "contains"

$$u = E^2 + B^2$$

energy per unit volume. This quanity of energy stored in the field, by its creation, also has a conservation law. If we define $S = E \times B$ as "the energy per unit time per unit area transported by the field", then we have that

$$\frac{\partial u}{\partial t} = -\nabla \cdot S$$

4.2 Momentum

A similar analysis can be done with respect to momentum. The gist is that you can calculuate the total force on a charge in some region, then using newtons second law $F = p'(t)$ you get an expression for the momentum. One of the terms then is interpreted as the momentum of the field itself, the other of the particules in the field. This is the quantity that is conserved.

4.3 Waves

Maxwells DE's in a vacume, that is no charges present simplify to a set of coupled first order equations. They can be decopled to give a set of second order equations namely that

$$\nabla^2 E = \frac{\partial^2 E}{\partial t^2}, \quad \nabla^2 B = \frac{\partial^2 B}{\partial t^2}$$

Thus the component vectors of E and B must satisfy these equations. The solutions of this DE are classically known and are given by waves. Note that the logic here is that if you satisfy Maxwells equations then you satisfy the wave equations (above) and therefore are a wave. It is not the case that any wave is a solution to Maxwells equations. In particular all solutions must be transverse waves and they must propogate at a given speed, that is the speed of light. We have systematically been ignoring the constants in these equations but it is the case that there are two electromagnetic constants ϵ, μ that are measured by moving charges near each other. Then it is miraculous that $1/\sqrt{\epsilon\mu} = c$ the speed of light. Finally the DE above implies that the speed of the wave that it describes is given by $1/\sqrt{\epsilon\mu}$.

Remark. At this point it is still not clear that there are actually non-trivial solutions allowed by Maxwells equations, but it is easy to check that a transverse wave would satisfy all of them. Thus the converse to the logic above can be checked explicitly.

Remark. Here the speed of the wave is given by the solution of the DE which is controlled by E and B. In media these change and therefore the speed also changes. Therefore it is a logical consequence of this setup that EM waves will physically slow down in matter.

5 The General Solution

In general we do not have that E is curl free and therefore our definition of V from electrostatics does not make sense. Recalling that B is always div free we have that the magnetic potential A provides the necissary adjustment to E, that is we have that

$$\nabla \times \left(E \frac{\partial A}{\partial t} \right) = 0.$$

Thus we have that there is some scalar field, which we call V, whose gradient is $E + \frac{\partial A}{\partial t}$. Giffiths points out that solving in terms of potentials then entails finding four functions, the three components of A and the one of V. The potentials however are not uniquely defined, that is many potentials will result in the required fields. Fixing certain conditions on the potentials is computationally useful and this is done while finding the general solution, which is

$$E(\mathbf{r}, t) = \int \frac{\rho(r', t_r)}{r^2}\hat{\mathbf{r}} + \frac{\dot{\rho}(r', t_r)}{cr}\hat{\mathbf{r}} - \frac{\dot{J}(r', t_r)}{c^2 r}\hat{\mathbf{r}} \quad \mathrm{d}r'$$

$$B(\mathbf{r}, t) = \int \left[\frac{J(r', t_r)}{r^2} + \frac{\dot{J}(r', t_r)}{cr} \right] \times \hat{\mathbf{r}} \quad \mathrm{d}r'$$

Here we have that

$$t_r \quad := \quad t - \frac{r}{c}.$$

This is called the retarded time. Recall that $r = |\mathbf{r} - r'|$. This is capturing the fact that the EM information is traveling only at the speed of light. Grifiths proof that this time retardation is necissary is that the *potentials* in the integrals satisfy Maxwells equations and the DE;s derived from them. This strikes me as a sufficiency proof. I didnt inspect the logic carefully but I guess it would be in principle possible to find if each of the steps were actually iff, that is satisfy the equation iff a solution and a solution iff using the retarded time. Right now, or as presnted the logic is that this form implies a solution to some DE's that were derived from Maxwells.

Griffiths writes

> (These) are the (causal) solutions to Maxwell's equations. For some reason, they do not seem to have been published until quite recently the earliest explicit statement of which I am aware was by Oleg Jefimenko, in 1966. In practice Jefimenko's equations are of limited utility, since it is typically easier to calculate the retarded potentials and differentiate them, rather than going directly to the fields. Nevertheless, they provide a satisfying sense of closure to the theory.

References

[Gri13] David J. Griffiths. *Introduction to electrodynamics*. Always learning. Pearson, Boston, 4. ed., international ed edition, 2013.